

# ROLLERCOASTER PERMUTATIONS AND PARTITION NUMBERS

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**ABSTRACT.** This paper explores the properties of partitions of roller coaster permutations. A roller coaster permutation is a permutation that alternates between increasing and decreasing a maximum number of times, while its subsequences also alternate between increasing and decreasing a maximum number of times simultaneously. The focus of this paper is on achieving an upper bound for the partition number of a roller coaster permutation of length  $n$ .

## 1. INTRODUCTION

Roller coaster permutations first show up in a work of Ahmed & Snevily [2] where roller coaster permutations are described as a permutations that maximize the total switches from ascending to descending (or visa versa) for a permutation and all of its subpermutations simultaneously. More basically, this counts the greatest number of ups and downs or increases and decreases for the permutation and all possible subpermutations. Several of the properties of roller coaster permutations that were conjectured by Ahmed & Snevily are proven in a paper of the first author [1] and are relied on heavily in developing an upper bound for the partition number of a roller coaster permutation.

These permutations are connected to pattern avoiding permutations as is seen in Mansour [5] in the context of avoiding the subpermutation 132. These are also strongly connected to forbidden subsequences and partitions of permutations is seen in Stankova [6], where certain forbidden subsequences end up being roller coaster permutations, particularly  $F(1, 1)$  is a subset of  $RC(n)$ . Consequently, these permutations are related to stack sortable permutations as seen in Egge & Mansour [3], where the connection between forbidden subsequences and stack sortability is made.

Kezdy, Snevily & Wang[4] explored partitions of permutations into increasing and decreasing subsequences, where they took the approach of associating a graph to a permutation, they then translated the notion of partitions to the lack of existence of certain subgraphs. Our approach here relies rather on the underlying structure of these permutations, particularly the alternating structure, together with the relative positions of entries that are forced on roller coaster permutations.

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## 2. BACKGROUND

**Definition 2.1.** A permutation of length  $n$  is an ordered rearrangement on the set  $\{1, 2, 3, \dots, n\}$  for some  $n$ . The collection of all such permutations is denoted  $S_n$ .

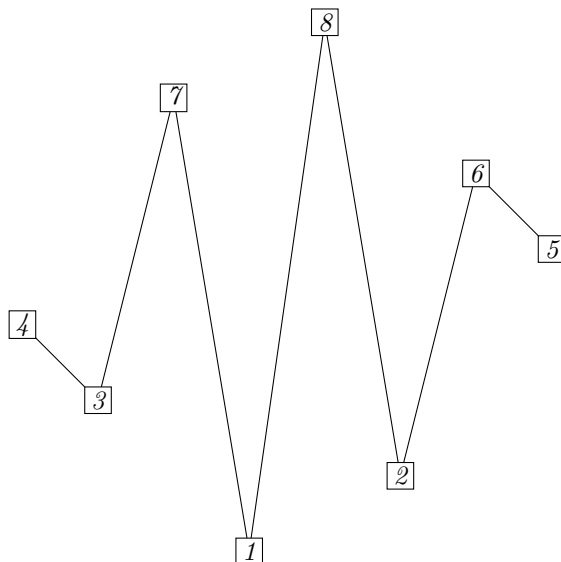
**Definition 2.2.** A Roller coaster permutation is a permutation that maximizes the number of changes from increasing to decreasing over itself, and all of its subsequences, simultaneously. Here a subsequence of a permutation is an ordered subset of the original permutation [2].

The collection of all roller coaster permutations in  $S_n$  is denoted  $RC_n$ . and have been explicitly found for small  $n$  and are as follows:

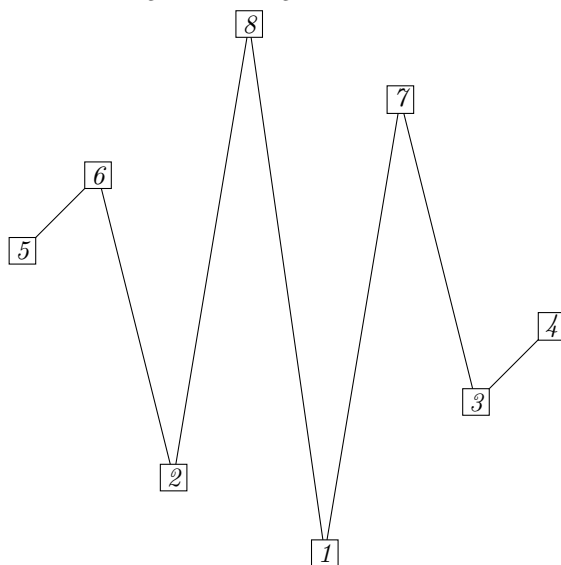
$RC(3) = \{132, 213, 231, 312\}$   
 $RC(4) = \{2143, 2413, 3142, 3412\}$   
 $RC(5) = \{24153, 25143, 31524, 32514, 34152, 35142, 41523, 42513\}$   
 $RC(6) = \{326154, 351624, 426153, 451623\}$   
 $RC(7) = \{3517264, 3527164, 3617254, 3627154, 4261735, 4271635, 4361725, 4371625, 4517263, 4527163, 4617253, 4627153, 5261734, 5271634, 5361724, 5371624\}$   
 $RC(8) = \{43718265, 46281735, 53718264, 56281734\}$   
 $RC(9) = \{471639285, 471936285, 472639185, 472936185, 481639275, 481936275, 482639175, 482936175, 528174936, 528471936, 529174836, 529471836, 538174926, 538471926, 539174826, 539471826, 571639284, 571936284, 572639184, 572936184, 581639274, 581936274, 582639174, 582936174, 628174935, 628471935, 629174835, 629471835, 638174925, 638471925, 639174825, 639471825\}$ . [2]

**Definition 2.3.** An alternating permutation is a permutation  $\pi$  of such that  $\pi_1 < \pi_2 > \pi_3 < \dots$  and a reverse alternating permutation is a permutation  $\pi$  of such that  $\pi_1 > \pi_2 < \pi_3 < \dots$

**Example 2.4.** The following is a graphical representation of the permutation  $\{4, 3, 7, 1, 8, 2, 6, 5\}$ . This permutation is reverse alternating, as you can see that the first entry is greater than the second entry and the pattern defined above continues throughout the entire permutation.



**Example 2.5.** The permutation  $\{5, 6, 2, 8, 1, 7, 3, 4\}$ , pictured below, is an example of a forward alternating permutation. Sometimes forward alternating permutations are simply referred to as being alternating.

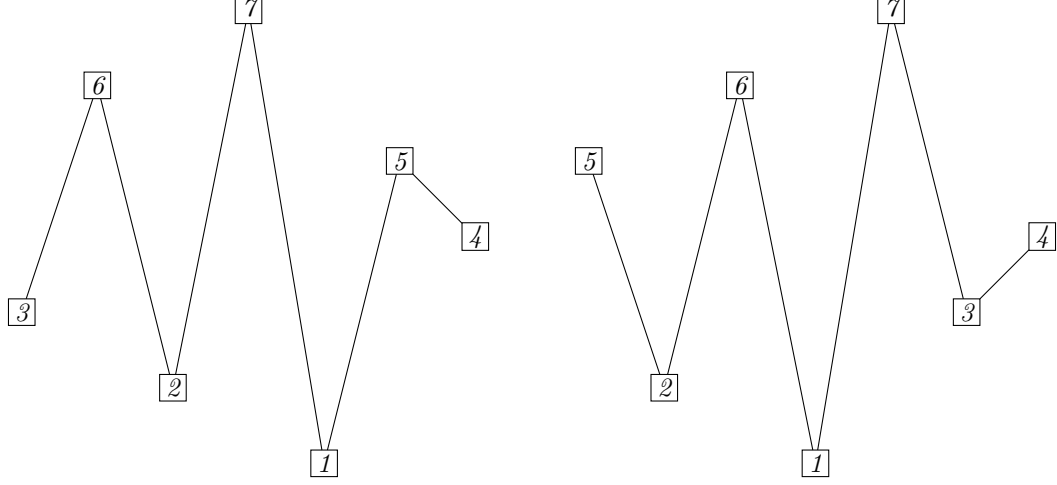


**Definition 2.6.** The reverse of a permutation  $\pi$  is the permutation with entries given by  $(\pi_n, \pi_{n-1}, \dots, \pi_1)$ .

**Definition 2.7.** The complement of a permutation  $\pi$  is  $(n+1-\pi_1, n+1-\pi_2, \dots, n+1-\pi_n)$ .

**Example 2.8.** An example of a permutation and its complement are  $\{3, 6, 2, 7, 1, 5, 4\}$  and  $\{5, 2, 6, 1, 7, 3, 4\}$ . These permutations follow the definition above, notice that the first element of each, 3 and 5 fit in the equation  $5 = 7 + 1 - 3$ . Both of these permutations have been graphically displayed below. The reverse of  $\{3, 6, 2, 7, 1, 5, 4\}$  is

$\{4,5,1,7,2,6,3\}$ . Notice that the reverse and complement of a permutation aren't necessarily equal.



Below we give a collection of theorems regarding the structure of roller coaster permutations. We will use these heavily in arriving at an upper bound for the partition number.

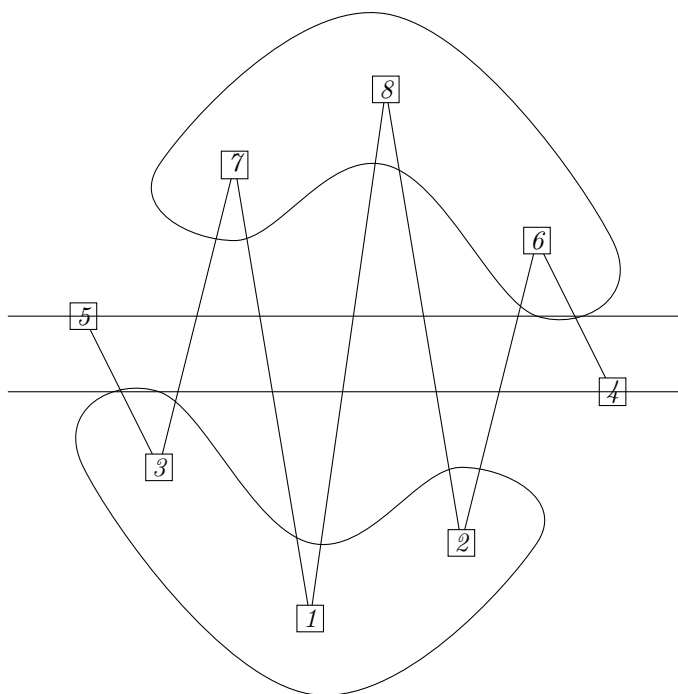
**Theorem 2.9.** *Given  $\pi \in RC_n$ , the reverse and complement of  $\pi$  are also members of  $RC_n$  [2].*

**Theorem 2.10.** *Given  $\pi \in RC_n$ , we have that  $\pi$  is either alternating or reverse alternating [1].*

**Theorem 2.11.** *Given  $\pi \in RC_n$ ,  $|\pi_1 - \pi_n| = 1$ , [1].*

**Theorem 2.12.** *For  $\pi \in RC_n$  if  $\pi$  is alternating then  $\pi_i > \pi_1, \pi_n$  for even  $i$ . If  $\pi$  is reverse alternating then  $\pi_i > \pi_1, \pi_n$  for odd  $i$  [1].*

**Example 2.13.** *Below is a graphical representation of the permutation  $\{5,3,7,1,8,2,6,4\}$ . As you can see, the end points are 5 and 4, which have a difference of 1 as stated in Theorem 2.8. Also in the drawing below, notice that some elements have been circled into different sets, these being 7,8 and 6 in the "top" set and 3,1 and 2 in the "bottom" set. Notice that the top set is entirely comprised of numbers greater than the end points and the bottom is comprised entirely of numbers less than the end points. The top set has elements that are in the odd indices while the bottom set has elements that are in the even indices, just as Theorem 2.9 states.*



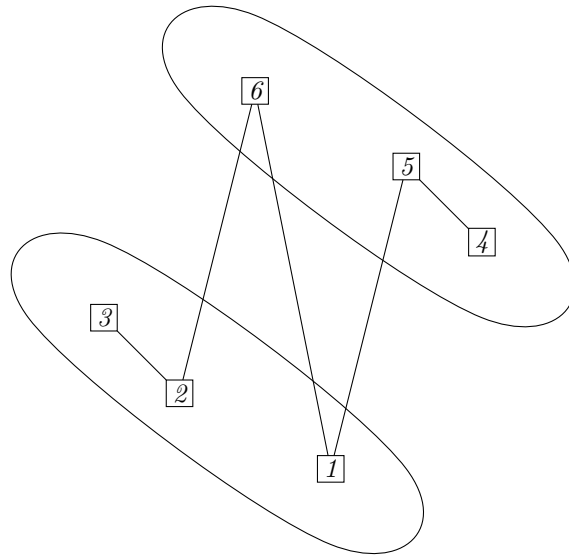
**Definition 2.14.** A subsequence of a permutation is said to be monotonic, if it is strictly increasing or strictly decreasing.

Monotonic subsequences are sometimes called runs. In the permutation  $\{5, 8, 2, 6, 3, 9, 1, 7, 4\}$  there are a few runs. The run  $(589)$  is an increasing, while  $(974)$  is a decreasing run. The longest run in this permutation is  $(8631)$ .

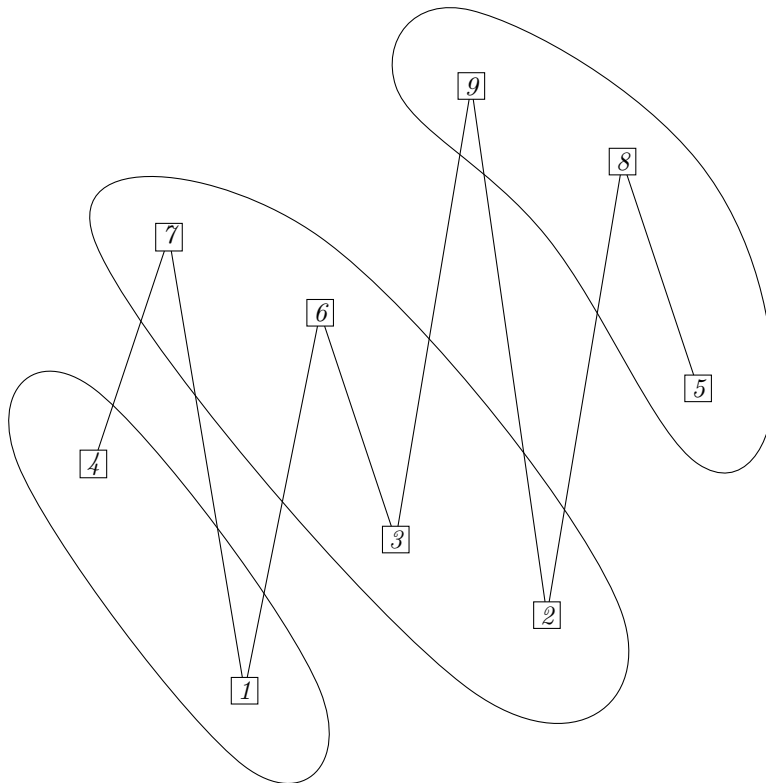
**Definition 2.15.** A partition of a permutation is the set of disjoint monotonic subsequences of that permutation.

**Definition 2.16.** The partition number of a permutation, denoted  $P(\pi)$ , is the least number of partitions that permutation  $\pi$  can be broken into.

**Example 2.17.** Here you can see a graphical representation of the permutation  $\{3, 2, 6, 1, 5, 4\}$ . The oval distinguish the runs in the partition. Notice that there are two ovals each with three numbers in them. This shows that the runs in this permutation are  $\{3, 2, 1\}$  and  $\{6, 5, 4\}$ . It also shows that there are two runs, which means that this permutation has a partition number of 2, or in other words,  $P(326154) = 2$ .



**Example 2.18.** Here is another partitioned permutation. This time the permutation is  $\{4, 7, 1, 6, 3, 9, 2, 8, 5\}$ .



**Definition 2.19.**  $P(n)$  is the number of partitions that any permutation  $\pi$  can be broken into where  $\pi \in RC_n$ .  $P_{max}(n)$  is the upper bound on  $P(n)$ .

### 3. RESULTS

**Theorem 3.1.** For  $\pi \in RC_n$  the partition number  $p_{max}(\pi)$  is bounded above by  $\lfloor \frac{\lceil \frac{n-2}{2} \rceil}{2} \rfloor + 2$ .

*Proof.* Without loss of generality we may assume that  $\pi \in RC_n$  and  $\pi$  is reverse alternating, i.e.  $\pi$  starts with a descent, otherwise we could take the compliment of  $\pi$  which is also in  $RC_n$ , since complimenting exchanges alternating for reverse alternating and we may then use the same argument that follows and then take the compliment again.

- Excluding the endpoints, there will be  $\lceil \frac{n-2}{2} \rceil$  positions below the endpoints and  $\lfloor \frac{n-2}{2} \rfloor$  positions above the endpoints. Those positions below the endpoints are at even indices and the positions above are at odd indices.
- Partition the even indices into contiguous increasing runs and do the same with odd indices. The number of runs made from even indices will be  $\lfloor \frac{\lceil \frac{n-2}{2} \rceil}{2} \rfloor + 1$
- Note that when partitioning a forward or reverse alternating partition into contiguous increasing runs, the  $k^{th}$  run will have an earliest start at index  $2k-2$  for  $k > 1$  and the latest finish for this run will be at index  $2k$ . The  $k^{th}$  index from the bottom partitions comes before the  $k^{th}$  partition of the top partitions due to  $\pi$  being reverse alternating. So the latest finish for the  $k-1^{st}$  run is, at worst, equal to the latest start for the  $k^{th}$  run, thus the  $k+1^{st}$  segment from the top starts after the  $k^{th}$  run from the bottom.
- So the first run on the top pairs with the start point and then the  $k^{th}$  run on the bottom pairs with the  $k+1^{st}$  run on the top. If the number of runs on the bottom is greater than the number of runs on the top then the second to last run on the bottom pairs with the end point and we have an extra  $+1$  in the partition number. Otherwise the last run on the bottom will pair with the end point. Thereby establishing the claim.

□

We found exact numbers for  $P_{max}(n)$  for  $n < 15$  experimentally using code developed in the Sage computer algebra system. These values can be found in the table below.

$n$	$P_{max(n)}$	$\lfloor \frac{\lceil \frac{n-2}{2} \rceil}{2} \rfloor + 2$
3	2	2
4	2	2
5	2	3
6	3	3
7	3	3
8	3	3
9	4	4
10	4	4
11	4	4
12	4	4
13	5	5
14	5	5

Note that the bound found in the theorem above is nearly sharp. For  $n < 15$  the upper bound we found is very close to the actual values of  $P_{max}(n)$ . The only deviation is our upper bound at  $n = 5$  was 1 greater than the actual value.

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